Gp1&3 Gp2

Q1: $\iint_D \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dA$ $\iint_D x^3\,dA$

$D: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $D: y=\sqrt{x},\ y=3\sqrt{x},\ xy=1,\ xy=4$

Ans: $\frac{2ab}{3}\pi$ $\frac{56}{9}$

Idea: ① Choose a good coordinate:

$x=ar\cos\theta \quad y=br\sin\theta$ $u=y^2/x \quad v=xy$

② Transform to the new coordinate & Calculating of Jacobian.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = ab\begin{vmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{vmatrix}$$

$$= abr$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix}-y^2/x^2 & 2y/x \\ y & x\end{vmatrix}$$

$$= -\frac{3y^2}{x} = -3u$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{3u}$$

③ Integral

$$= \int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\,|abr|\,dr\,d\theta$$

$$= 2\pi\cdot ab\cdot\frac{1}{3}(1-r^2)^{\frac{3}{2}}\Big|_0^1$$

$$= \frac{2\pi}{3}ab$$

Integral

$$= \int_1^4\int_1^9 \frac{v^2}{u}\left|-\frac{1}{3u}\right|\,du\,dv$$

$$= -\frac{1}{3u}\Big|_1^9 \cdot \frac{1}{3}v^3\Big|_1^4$$

$$= 56/9$$

Q2 ① $\iiint_\Delta x\,dV$

Δ tetrahedron bounded by (0,0,0) (1,0,0), (0,2,0) (0,0,3)

② $\iiint_\Delta z\,dV$

Δ: tetrahedron with vertices at (0,0,0), (2,0,0), (0,1,0) (0,0,4)

③ $\iiint_\Omega xyz\,dV$

Ω: $x^2+y^2+z^2 = R^2$

$x, y, z \geq 0$

①:

find the eqn of this plane

- $(0,0,3) - (1,0,0) = (-1,0,3)$
- $(0,2,0) - (1,0,0) = (-1,2,0)$
- $(x,y,z) - (1,0,0) = (-1+x, y, z)$

eqn: $\begin{vmatrix} x-1 & y & z \\ -1 & 0 & 3 \\ -1 & 2 & 0 \end{vmatrix} = 0$

$$6x+3y+2z = 6$$

← An alternative way to do it is use the substitution $x=u$, $y=2v$, $z=3w$

$\Rightarrow$ eqn becomes $u+v+w=1$

$$\text{Integral} = \int_0^1 \int_0^{2-2x} \int_0^{\frac{1}{2}(6-6x-3y)} x \,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{2-2x} x\left[3(1-x) - \tfrac{3}{2}y\right] dy\,dx$$

$$= \int_0^1 x\left(6(1-x)^2 - 3(1-x)^2\right) dx$$

$$= \int_0^1 3x(1-x)^2\,dx$$

$$= \int_0^1 3x - 6x^2 + 3x^3\,dx$$

$$= \frac{3}{2} - 2 + \frac{3}{4} = \frac{1}{4}$$

Gp2 Ans: $\frac{4}{3}$

3. $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^R r^5 \cos\varphi \sin^3\varphi \cos\theta \sin\theta \,dr\,d\varphi\,d\theta$

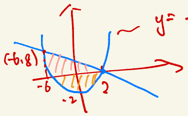
$= \left[\frac{1}{6}r^5\right]_0^R \cdot \left[\frac{1}{4}\sin^4\varphi\right]_0^{\pi/2} \cdot \left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}$

$= \frac{1}{48}R^6$

I&II

Q3. $\int_{-6}^{2}\int_{x^2/4-1}^{2-x} f\,dy\,dx$

change to dxdy



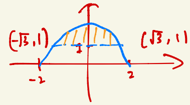
$y=\frac{x^2}{4}-1$

$x=\pm 2\sqrt{y+1}$

$$\int_{-2}^{0}\int_{-2\sqrt{y+1}}^{2\sqrt{y+1}} f\,dx\,dy + \int_{0}^{8}\int_{-2\sqrt{y+1}}^{2-y} f\,dx\,dy$$

$\int_{-2}^{2}\int_{1}^{\max\{1,4-x^2\}} f\,dy\,dx$

change to dxdy



$$\int_{1}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}} f\,dx\,dy$$

$$\left(\neq\ 2\int_{1}^{4}\int_{0}^{\sqrt{4-y}} f\,dx\,dy\right)$$

I III VI

Q4 Find the area bounded by
$(x^2+y^2)^2 = 2a^2(x^2-y^2) \quad a>0$

$(x^2+y^2)^2 = 25(x^2-y^2)$

Polar coordinate:

$r^4 = 2ar^2(\cos^2\theta - \sin^2\theta)$

$r^2 = 2a\cos 2\theta$

$\cos 2\theta \geq 0$ for $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ and $\frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}$

$$\text{Area} = 2\int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{2a\cos 2\theta}} r\,dr\,d\theta$$

$$= 2a^2\int_{-\pi/4}^{\pi/4}\cos 2\theta\,d\theta$$

$$= 2a^2$$

⋮

25

Find the area of the region common to
$r = 1-\cos\theta$ and $r = 1+\cos\theta$

$1-\cos\theta \leq 1+\cos\theta$ for θ in quadrant I & IV

$1+\cos\theta \leq 1-\cos\theta$ for θ in quadrant II & III

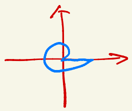
$$\text{Area} = \int_{\text{quadrant I\&IV}}\int_0^{1-\cos\theta} r\,dr\,d\theta$$

$$+ \int_{\text{quadrant II\&III}}\int_0^{1+\cos\theta} r\,dr\,d\theta$$

$$= \frac{3\pi}{2} - 4$$

Q5 Gp II

S bounded by $r=\theta$ and $[0,2\pi]$ on x-axis



Evaluate $\iint_S \log(x^2+y^2)\,dA$

$$= \lim_{c\to 0^+} \int_c^{2\pi}\int_c^{\theta} \log(r^2)\, r\,dr\,d\theta$$

$$\int_c^{\theta} \log(r^2) r\,dr = r^2\log r - \frac{1}{2}r^2 \Big|_c^{\theta}$$

$$\int_c^{2\pi}\int_c^{\theta} \log(r^2) r\,dr\,d\theta = \int_c^{2\pi} \theta^2\log\theta - \frac{1}{2}\theta^2 - c^2\log c + \frac{1}{2}c^2 \, d\theta$$

$$= \frac{1}{3}\theta^3\log\theta - \frac{5}{18}\theta^3 + \left(\frac{1}{2}c^2 - c^2\log c\right)\theta \Big|_c^{2\pi}$$

$$= \frac{8}{3}\pi^3\log 2\pi - \frac{20\pi^3}{9} - \frac{1}{3}c^3\log c + \frac{1}{9}c^3$$

$$+ \left(\frac{1}{2}c^2 - c^2\log c\right)(2\pi - c)$$

$$\to \frac{8}{3}\pi^3\log 2\pi - \frac{20\pi^3}{9} \quad \text{as} \quad c\to 0^+$$

(Some common mistakes: As $\int_0^{\theta}\log(r^2)r\,dr = \lim_{c\to 0^+}\int_c^{\theta}\log(r^2)r\,dr$

$$= \frac{1}{3}\theta^3\log\theta - \frac{1}{9}\theta^3,$$

So $\int_0^{2\pi}\int_0^{\theta}\log(r^2)r\,dr = \int_0^{2\pi}\left(\frac{1}{3}\theta^3\log\theta - \frac{1}{9}\theta^3\right)d\theta$)

This is not correct because $\log(r^2)r$ is not continuous at the origin, so we cannot apply Fubini theorem to conclude $\iint_R \log(r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^{r}\log(r^2) r\,dr\,d\theta$.

A counter-example to Fubini's thm.

$$f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\int_0^1 \int_0^1 f(x,y)\, dx dy = \int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\, dx dy$$

$$= \int_0^1 -\frac{x}{x^2+y^2}\Big|_{x=0}^{x=1}\, dy$$

$$= \int_0^1 -\frac{1}{1+y^2}\, dy$$

$$= -\tan^{-1} y \Big|_0^1 = -\frac{\pi}{4}$$

while $\displaystyle\int_0^1 \int_0^1 f(x,y)\, dy dx = \int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\, dy dx$

$$= \int_0^1 \frac{y}{x^2+y^2}\Big|_{y=0}^{y=1}\, dx$$

$$= \int_0^1 \frac{1}{1+x^2}\, dx$$

$$= \tan^{-1} x \Big|_0^1 = \frac{\pi}{4}$$

P - Parallelotope formed by $x+y-z=\pm 1$, $x-2y=\pm 2$, $x+z=\pm \pi/2$ — Gp I Q8, G II, III, Q8 is similar

Evaluate $\iiint_P \cos(x+z)\,dV$

Let $u = x+y-z$, $v = x-2y$, $w = x+z$ $\Rightarrow \dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -5$

$$\Rightarrow \text{Integral} = \frac{1}{5}\int_{-\frac{\pi}{2}}^{\pi/2}\int_{-2}^{2}\int_{-1}^{1} \cos w \, du\,dv\,dw$$

$$= \frac{1}{5}\cdot 2\cdot 4\cdot 2 = \frac{16}{5}$$

Theorem: Let Ω be a region in $\mathbb{R}^2$ (or $\mathbb{R}^3$), and $\vec{F}: \Omega \to \mathbb{R}^3$ a C^∞ vector field

The followings are equivalent

1) $\vec{F}$ is conservative. i.e. there exists a function $f: \Omega \to \mathbb{R}$ with $\vec{F} = \nabla f$
2) for any piecewise smooth closed curve C in Ω, then

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

3) If C_1, C_2 are two piecewise smooth curve in Ω, with the same initial point and end point, then

$$\int_{C_1} \vec{F} d\vec{r} = \int_{C_2} \vec{F} \cdot dr$$

Pf: Lecture 17

Thm (Green) C simple closed, piecewise regular, anti-clockwise
D is the region bounded by C

Then $$\oint_C P dx + Q dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Q1 : Show that the integral $\int_A^B z^2dx + 2ydy + 2xzdz$

is independent of the choice of the curve from A to B.

Ideas : Try to show that the vector field

$$\vec{F} = (z^2, 2y, 2xz)$$

is conservative.

i.e. we ought to find f with $\nabla f = \vec{F}$

$f_x = z^2 \Rightarrow f = xz^2 + g(y,z)$

$f_y = 2y \Rightarrow \quad g_y = 2y \quad \Rightarrow g = y^2 + h(z)$

i.e. $f = xz^2 + y^2 + h(z)$

$f_z = 2xz \Rightarrow 2xz + h_z = 2xz$

we may just take $h \equiv 0$

so $f = xz^2 + y^2$

Ans : Consider $f = xz^2 + y^2$. note that $\nabla f = (z^2, 2y, 2xz)$

so $\int_A^B z^2dx + 2ydy + 2xzdz$ is independent of path

Q2 a) Let C & D be as in the Green's thm.

Show that $\text{Area}(D) = \int_C x\,dy = -\int_C y\,dx$

b) Show that in polar coordinates

$$\text{Area}(D) = \frac{1}{2}\int_C r^2\,d\theta$$

Ans: a) $\int_C x\,dy = \int_C 0\cdot dx + x\cdot dy \overset{\text{Green}}{=} \iint_D \frac{\partial x}{\partial x} - \frac{\partial 0}{\partial y}\,dA = \iint_D dA = \text{Area}(D)$

Similarly $-\int_C y\,dx = \iint_D \frac{\partial 0}{\partial x} - \frac{\partial(-y)}{\partial y}\,dA = \iint_D dA = \text{Area}(D)$

b) By a) $\text{Area}(D) = \int_C x\,dy = \int_C r\cos\theta\,d r\sin\theta = \int_C r\cos\theta\sin\theta\,dr + \int_C r^2\cos^2\theta\,d\theta$

$$+\quad \text{Area}(D) = -\int_C y\,dx = -\int_C r\sin\theta\,d\,r\cos\theta = -\int_C r\sin\theta\cos\theta\,dr + \int_C r^2\sin^2\theta\,d\theta$$

$$2\,\text{Area}(D) = \int_C r^2\,d\theta$$

Q3 : find the area bounded by the polar curve

$r^2 = 25\cos 2\theta \qquad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$

Ans: $\text{Area} = \frac{1}{2}\int_C r^2 \, d\theta$

Step 1: Choose a parametrization

$(r, \theta) = (5\sqrt{\cos 2t}, t) \qquad -\frac{\pi}{4} \le t \le \frac{\pi}{4}$

$\text{Area} = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 25\cos 2t \, dt$

$= \frac{25}{2}$